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RESEARCH TITLE

ANALYSIS OF HOMEOMORPHISM GROUPS FOR FINITE SPACES WITH SMALL CARDINALITY USING BURNSIDE'S THEOREM AND THE SEMIDIRECT PRODUCT

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Abstract

In this article, I compute the number of orbit spaces for finite spaces $|X| \leq 4$ using Burnside's Counting Theorem. The group Homeo(X), consisting of all homeomorphisms from a space X onto itself for $|X| \leq 4$,is calculated using two different methods that yield identical results. These findings contribute to the understanding of the algebraic structure of finite spaces and their applications in mathematics (Elmsmary, 2016).

Key Words: homeomorphism group, split exact sequence ,semi direct product, Burnside's counting theorem, finite topological spaces.

عنوان البحث

تحليل زمر التماثلات التماثلية للمساحات الطوبولوجية المنتهية ذات الحجم الصغير باستخدام مبرهنة بيرنسايد والطريقه شبه المباشر

المستخلص

في هذا البحث، نقوم بحساب عدد فضاءات المدارات للمساحات الطوبولوجية المنتهية $|X| \leq 4$ باستخدام مبرهنة بيرنسايد للعد. بالإضافة إلى ذلك، نحسب زمرة التماثلات التماثلية Homeo(X)، التي تمثل جميع التماثلات التماثلية للمساحة على نفسها، عندما يكون حجمها تم استخدام طريقتين مختلفتين في الحساب، وكلتاهما أعطت نفس النتائج. هذه النتائج تساهم في تعزيز الفهم حول البنية الجبرية للمساحات المتناهية، وتوضيح تطبيقاتها في الرياضيات

Introduction:

The object of this paper is to consider finite space, i.e. space having only a finite number of points, through which we define orbit set and count it using **Burnside Counting (**Dummit & Foote, 1999).

The space X/G is topologized by identification .Where $X/_G$ is called the orbit space (Kelley, 2008).

Let X be a space.$Homeo(X)$ is the group of all homeomorphism form X to itself, defined as $Homeo(X) = \{h|h : X \to X \text{ is a homeomorphism}\}$ (Kono & Ushitaki, 2003).

In this paper, $Homeo(X)$ has been computed for a finite space with small cardinality (Kosnowski, 1980).

Then $Homeo(X)$ is a group with respect to composition (Elmsmary, 2016).

$Homeo(X$ can be made a topological group by using compact-open topology (Kelley, 2008).

Both methods of computing $Homeo(X)$ yield the same result.

Although the direct method is easier in computations the other method is more important in theoretical aspects (Kono & Ushitaki, 2003) of the subject.

Through this research paper can calculate $Homeo(X)$, if it a space contains a large number of elements (Kosnowski, 1980).

1. Homeomorphism Groups ($\boldsymbol{Homeo(X)}$) and Group Actions:

Theorem 1 (Elmsmary, 2016):

$Homeo(X)$ is a $group$ under composition .

Remark:

Homeo(X) is called homeomorphism group of a space X.

Example (1):

Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$.

To find Homeo(X):

There are $3! = 6$ one to one onto function from X to X.

The hoCmeomorphism are only id and h, where

$id(a) = a$ $\quad$ $id(b) = b$ $\quad$ $id(c) = c$

$h(a) = a$ $\quad$ $h(b) = c$ $\quad$ $h(c) = b$

Then $Homeo(X)$=$\{id, h\}$

Then Homeo(X $\cong \mathbb{Z}_2$, the cyclic group of order 2.

Group Actions (Dummit & Foote, 1999):

A group G acts on a nonempty a set **S** , if there is a mapping :

Denoted by $\varphi(g, x) \equiv g.x$ for any $g \in G$ and any $x \in S$ $\varphi: G \times S \to S$ such that :

i) $e.x = x$. Where e is the identity of G .

ii) $g.(h.x) = (gh).x$, for any $g, h \in G$ and any $x \in S$.

S is thus called a $\boldsymbol{-Set}$ **.**

Let **S** be a nonempty set and $G = \{f | f : S \to S \ is\ 1-1\ and\ onto\ function\}$.

The group G acts on the set as follows :

$g.x = g(x)$ for any $g \in G$ and any $x \in S$.

Define a binary relation $\sim$ on **S** by $x \sim y$ if $g.x = y$ .Let S be a $\boldsymbol{G-Set}$.

For some $\in G$.$\sim$ is an equivalence relation on S .The equivalence class of $x \in S$ is as follows:

$[x]$=$\{y \in S: y \sim x\} = \{y: g.x = y\ for\ some\ g \in G\} = \{g.x: g \in G\} \equiv G.x$, which is called the orbit of x .The orbit set is the quotient set.

$S/_G = \{G.x: x \in S\}$, the set of all orbits of the elements of S.

The projection is given by $\rho: S \to S/_G$, where $\rho(x) = G.x$ it is onto.

Example (2):

Let $S = \{a, b, c, d\}$ and the group of transformations of S is:

$$G = \left\{\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & b & d & c \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}\right\}.$$

$[a]$ =The orbits are :

$G.a = \{a, b\}, [b] = G.b = \{a, b\}$ and also $[c] = [d] = \{c, d\}$.

The orbit set $S/_G = \{[a], [c]\}$.It has two orbits .

Theorem(Burnside Counting Theorem)(Dummit & Foote, 1999):

Let S be a $\boldsymbol{G-Set}$ .For $g \in G$,let $S^g = \{x \in S: g.x = x\}$, the set of element fixed by g. Then the number of orbits is given by :

$|S/_G| = \frac{1}{|G|} \sum_{g \in G} |S^g|$.

Example (3):

Consider the previous example .

$S = \{a, b, c, d\}$.Denote the elements of the group G by $id, \alpha, \beta, \delta$ respectively.

$S^{id} = \{a, b, c, d\}, S^{\alpha} = \{c, d\}, S^{\beta} = \{a, b\}, S^{\delta} = \emptyset. By$ **Burnside**:

$|S/_G| = \frac{1}{|G|} \sum_{g \in G} |S^g| = \frac{1}{4}(4 + 2 + 2 + 0) = 2$ orbits.

2.Topological Group and Orbit Spaces:

G is a topological group ,if

1- G is a group .

2- G has a topology ,and

3- the following two function are continuous the multiplication $\mu: G \times G \to G$ given by :

$\mu(x, y) = xy$ for any $x, y \in G$ (Kelley, 2008). .

Example (4):

$\mathbb{R}$ is a topological group with respect to addition and usual topology.

Orbit Spaces:

Let X be a $\boldsymbol{G-Set,}$ define a relation $\sim$ on X as follows:

$x \sim y$ iff there exists $g \in G$ such that $g.x = y$.

$\sim$ is an equivalence relation on X.

$[X] = \{g \in X,\ x \sim y\} = \{y \in Y: gx = gy \in G\}$.

$= \{g.x: g \in G\} \equiv GX$, the orbit space of X.

$X/_G = \{G.x: x \in X\}$, the set of all orbit and $\rho: X \to X/_G$.

$\rho(x) = G.x$ the projection (onto).

$X/_G$ is topologized by identification $X/_G$.

$X/_G$ is called the orbit space (Kelley, 2008).

Example (5):

$X = \{a, b, c\}$ $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$

$f(a) = a,\ f(b) = c,\ f(c) = b$.

$G = Homeo(X) = \{id, f\} \cong \mathbb{Z}_2$.

$[a] = \{g.a: g \in G\} = \{a\}$.

$[b] = \{g.b: g \in G\} = \{b, c\}$.

$[c] = \{g.c: g \in G\} = \{c, b\}$

Then $X/_G = \{[a], [b]\}$, the orbit space.

The topology on $X/_G$ is $\hat{\tau} = \{\emptyset,\ \{[a]\}, \{[b]\}, X/_G\}$.

3. Computation of $\boldsymbol{Homeo(X)}$:

Method (1):

In this method we compute the homeomorphism group $Homeo(X)$ of a finite space X. We consider the non equivalent topologies only. These are listed in(Elmsmary (2016)).

Example(6) :

Take $X = \{a, b, c, d\}$ and with topology $\tau = \{\emptyset, \{a, b\}, X\}$

$id(a) = a$ $id(b) = b$ $id(c) = c$ $id(d) = d$

$f(a) = b$ $f(b) = a$ $f(c) = c$ $f(d) = d$

$g(a) = a$ $g(b) = b$ $g(c) = d$ $g(d) = c$

$h(a) = b$ $h(b) = a$ $h(c) = d$ $g(d) = c$

The orders of $id, f, g, h \leq 2$ and $|Homeo(X)| = 4$.

Then $Homeo(X) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Method (2):

Let X=$\{x_1, x_2, \dots \dots, x_n\}$ be a finite space , and

U_i is minimal open set containing x_1. i.e it is the intersection of all open sets containing x_i.Define the equivalence relation $\sim$ on X as follows:

$x_i \sim x_j$ if $U_i \sim U_j$ .Take $\hat{X} = X/\sim$.

$\hat{X} = \{[X]: x \in X\}$ and $V_x: X \to \hat{X}$,the projection map (Kono & Ushitaki, 2003).

$V_x(x_i) = U_i \cup C_i$, where C_i is the smallest closed set containing x_i .Define

$Homeo_x(\hat{X}) \leq Homeo(\hat{X})$.

The main result is as follows:

For a finite space X, the following is a split exact sequence:

$1 \to \prod_{[x]\in \hat{X}} Homeo([x]) \xrightarrow{i} Homeo(X) \to Homeo_x(\hat{X}) \to 1$ (Kono & Ushitaki, 2003)

This can be written in terms of a semidirect product as follows:

$Homeo(X) \cong \left(\prod_{[x]\in \hat{X}} Homeo([x])\right) \rtimes Homeo_x(\hat{X})$. (Elmsmary, 2016)

Example(7):

Take $X = \{a, b, c, d\}$ and with topology $\tau = \{\emptyset, \{a, b\}, X\}$

$U_a = X$	$C_a = \{a, b\}$
$U_b = X$	$C_b = \{a, b\}$
$U_c = \{c, d\}$	$C_C = X$
$U_d = \{c, d\}$	$C_d = X$

$V_x([a]) = U_a \cap C_a = \{a, b\} = [a, b]$

$V_x([b]) = U_b \cap C_b = \{a, b\} = [a, b]$

$V_x([c]) = U_c \cap C_c = \{c, d\} = [c, d]$

$V_x([d]) = U_d \cap C_d = \{c, d\} = [c, d]$

$Homeo_x(X) = \{f \in Homeo(X): \# f([x]) = \# [x]\} \cong \{id\}$.

$\prod_{[x]\in X} Homeo(X) = Homeo([a]) \times Homeo([c])$.

$[a] = \{a, b\} \subseteq X$, relative topology $\tau = \{\emptyset, \{[a, b]\}\}$

$[c] = \{c, d\} \subseteq X$, relative topology $\tau = \{\emptyset, \{[c, d]\}\}$

$Homeo([a]) = \mathbb{Z}_2$

$Homeo([c]) = \mathbb{Z}_2$

$\prod_{[x]\in X} Homeo(X) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Then $Homeo(X) = \{id\} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$

$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

The groups $Homeo(X)$ for non equivalent spaces X with $|X| \leq 4$ have been computed by the two methods.

The results and number of orbits are given by the following table .

$\lvert X \rvert$	Non-equivalent Topological spaces	$Homeo(X)$	Number of orbits $\lvert X/_G \rvert$
1	$\tau = \{\emptyset, X\}$	$\{id\}$	1
2	$\tau_1 = \{\emptyset, X\}$	$\mathbb{Z}_2$	1
2	$\tau_2 = \{\emptyset, \{a\}, X\}$	$\{id\}$	1
2	$\tau_3 = \{\emptyset, \{a\}, \{b\}, X\}$	$\mathbb{Z}_2$	1
3	$\tau_1 = \{\emptyset, X\}$	S_3	1
3	$\tau_2 = \{\emptyset, \{b,c\}, X\}$	$\mathbb{Z}_2$	2
3	$\tau_3 = \{\emptyset, \{c\}, X\}$	$\mathbb{Z}_2$	2
3	$\tau_4 = \{\emptyset, \{c\}, \{c,b\}, X\}$	$\{id\}$	3
3	$\tau_5 = \{\emptyset, \{c\}, \{a,b\}, X\}$	$\mathbb{Z}_2$	2
3	$\tau_6 = \{\emptyset, \{c\}, \{a,c\}, \{b,c\}, X\}$	$\mathbb{Z}_2$	2
3	$\tau_7 = \{\emptyset, \{c\}, \{b\}, \{b,c\}, X\}$	$\mathbb{Z}_2$	2
3	$\tau_8 = \{\emptyset, \{c\}, \{b\}, \{a,c\}, \{b,c\}, X\}$	$\mathbb{Z}_2$	2
3	$\tau_9 = \{\emptyset, \{a\}, \{c\}, \{b\}, \{a,c\}, \{b,c\}, \{a,b\}, X\}$	S_4	1
4	$\tau_1 = \{\emptyset, X\}$	S_3	1
4	$\tau_2 = \{\emptyset, \{b,c,d\}, X\}$	$\mathbb{Z}_6$	2
4	$\tau_3 = \{\emptyset, \{c,d\}, X\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2
4	$\tau_4 = \{\emptyset, \{d\}, X\}$	$\mathbb{Z}_6$	2
4	$\tau_5 = \{\emptyset, \{c,d\}, \{b,c,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_6 = \{\emptyset, \{b,c\}, \{a,d\}, X\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	1
4	$\tau_7 = \{\emptyset, \{d\}, \{b,c,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_8 = \{\emptyset, \{d\}, \{a,b,c\}, X\}$	$\mathbb{Z}_6$	2
4	$\tau_9 = \{\emptyset, \{d\}, \{c,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{10} = \{\emptyset, \{c,d\}, \{b,c,d\}, \{a,c,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{11} = \{\emptyset, \{d\}, \{c,d\}, \{b,c,d\}, X\}$	$\{id\}$	1
4	$\tau_{12} = \{\emptyset, \{d\}, \{c,d\}, \{a,b,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{13} = \{\emptyset, \{d\}, \{b,c\}, \{b,c,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{14} = \{\emptyset, \{c\}, \{d\}, \{c,d\}, X\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2
4	$\tau_{15} = \{\emptyset, \{d\}, \{c,d\}, \{b,c,d\}, \{a,c,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{16} = \{\emptyset, \{d\}, \{c,d\}, \{b,d\}, \{b,c,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{17} = \{\emptyset, \{d\}, \{b,c\}, \{b,c,d\}, \{a,c,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{18} = \{\emptyset, \{d\}, \{b,c\}, \{a,d\}, \{b,c,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{19} = \{\emptyset, \{c\}, \{d\}, \{c,d\}, \{b,c,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{20} = \{\emptyset, \{c\}, \{d\}, \{c,d\}, \{a,b,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{21} = \{\emptyset, \{d\}, \{c,d\}, \{b,d\}, \{b,c,d\}, \{a,c,d\}, X\}$	$\{id\}$	4
4	$\tau_{22} = \{\emptyset, \{d\}, \{c\}, \{c,d\}, \{b,c,d\}, \{a,c,d\}, X\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2
4	$\tau_{23} = \{\emptyset, \{d\}, \{c\}, \{c,d\}, \{b,d\}, \{b,c,d\}, X\}$	$\{id\}$	4
4	$\tau_{24} = \{\emptyset, \{d\}, \{c\}, \{c,d\}, \{b,d\}, \{b,c,d\}, \{a,c,d\}, X\}$	$\{id\}$	4
4	$\tau_{25} = \{\emptyset, \{d\}, \{c\}, \{c,d\}, \{b,d\}, \{b,c,d\}, \{a,b,d\}, X\}$	$\{id\}$	4
4	$\tau_{26} = \{\emptyset, \{d\}, \{c\}, \{c,d\}, \{a,d\}, \{a,b,d\}, \{a,b,c\}, X\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2
4	$\tau_{27} = \{\emptyset, \{d\}, \{b,d\}, \{c,d\}, \{a,d\}, \{b,c,d\}, \{a,c,d\}, \{a,b,d\}, X\}$	$\mathbb{Z}_6$	2
4	$\tau_{28} = \{\emptyset, \{d\}, \{c\}, \{c,d\}, \{b,c\}, \{a,d\}, \{b,c,d\}, \{a,c,d\}, X\}$	$\mathbb{Z}_2$	2
4	$\tau_{29} = \{\emptyset, \{d\}, \{c\}, \{b\}, \{c,d\}, \{b,d\}, \{b,c\}, \{b,c,d\}, X\}$	$\mathbb{Z}_6$	2

4	$\tau_{30} = \{\emptyset, \{d\}, \{c\}, \{c,d\}, \{b,d\}, \{a,d\}, \{b,c,d\}, \{a,c,d\}, \{a,b,d\},$	$\mathbb{Z}_2$	3
4	$\tau_{31} = \{\emptyset, \{d\}, \{c\}, \{b\}, \{c,d\}, \{b,c\}, \{b,c,d\}, \{a,c,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{32} = \{\emptyset, \{d\}, \{c\}, \{b\}, \{b,d\}, \{c,d\}, \{a,d\}, \{b,c\}, \{b,c,d\}, \{a,b,d\}, X\}$	$\mathbb{Z}_2$	3
4	$\tau_{33} = \left\{ \begin{matrix} \emptyset, \{d\}, \{a\}, \{c\}, \{b\}, \{a,b\}, \{a,c\}, \{b,d\}, \{c,d\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \\ \{a,c,d\}, \{b,c,d\}, \{a,b,d\}, X \end{matrix} \right\}$	S_4	1

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